THE PLANE CONTACT PROBLEM OF THE THEORY OF CREEP

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The present paper presents a solution of the plane contact problem of the theory of creep, taking into account aging and the change of the modulus of the instantaneous deformation of the material.

This problem was studied in the linear formulation by Prokopovich [1].

In solving contact problems under the conditions of nonlinear creep, it is necessary to initiate the study from certain sufficiently well founded physical hypotheses regarding the relation between stress and deformations. From this point of view we do not consider it possible to use as a theory of creep one of the theories of aging*, since it may lead to incorrect results in solving these problems. As a fundamental physical hypothesis we here admit the theory of plastic heredity, advanced by Rabotnov [2], and developed by the author [3] for aging material.

A series of experimental investigations [4,5,6], completed recently and carried out especially to verify the basic equations of the theory of plastic heredity, which confirms a sufficiently good agreement of results obtained on the basis of this theory, with data from creep experiments for such materials as aluminum alloys, copper, low carbon steel and others.

In Section 1 the basic equations of the theory of plastic heredity are presented, which relate the components of deformation and stress, taking into account creep of the material in the case of plane state of strain.

Using these equations for a power law relating stresses and deformations, Section 2 presents a preliminary solution, directly in terms of stress, of the problem of equilibrium of a half-plane, under the condi-

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* We note that the theory of aging presented in various publications devoted to problems of creep is not related to the phenomenon of aging of materials.

tions of nonlinear creep subjected to a concentrated force applied normally to its free boundary.

Further, using this solution, it is proved in Section 3 that the solution of the plane contact problem of the nonlinear theory of creep reduces to a simultaneous solution of two coupled integral equations.

A discussion and solution of these equations are presented in subsections 2°-4° of this same section, for both cases of symmetric and antisymmetric loading of compressed bodies.

1. Relation between deformations and stresses for nonlinear creep. In the general case of a three-dimensional state of stress, the equations of the theory of plastic heredity which relate the strain intensity $\epsilon_i(t)$ with the stress intensity $\sigma_i(t)$, taking creep of the material into account, are of the following form:

$$\varphi^*[\varepsilon_i(t)] = \varphi[\varepsilon_i(t)]\,\varepsilon_i(t) = \sigma_i(t) - \int_{\tau_1}^{t} \sigma_i(\tau)\,\frac{\partial C(t,\tau)}{\partial \tau}\,d\tau \qquad (1.1)$$

Here $C(t, \tau)$ is the measure of creep of the material and $\phi^*[\epsilon_i(t)]$ is some function which characterizes the nonlinear relationship between stresses and deformations: both of these functions are determined from an experiment of simple creep: τ_1 is the age of the material, t is the time.

It is assumed

$$\varepsilon_i(t) = \frac{1}{\sqrt{6}}\sqrt{[\varepsilon_\theta(t) - \varepsilon_z(t)]^2 + [\varepsilon_z(t) - \varepsilon_r(t)]^2 + [\varepsilon_r(t) - \varepsilon_\theta(t)]^2 + 6\gamma_{r\theta}^2(t)} \qquad (1.2)$$

$$\sigma_i(t) = \frac{1}{\sqrt{6}}\sqrt{[\sigma_\theta(t) - \sigma_z(t)]^2 + [\sigma_z(t) - \sigma_r(t)]^2 + [\sigma_r(t) - \sigma_\theta(t)]^2 + 6\tau_{r\theta}^2(t)}$$

Using the usual transformation formulas relating the components of stress and strain in cylindrical coordinates (r, θ, z) with the corresponding components in a system of principal axes, and assuming that the stress and strain deviators have equal principal directions at any instant of time t, it follows from (1.1) for the case of plane state of strain*

$$\varepsilon_r(t)\,\varphi[\varepsilon_i(t)] = [\sigma_r(t) - \sigma(t)] - \int_{\tau_1}^{t} [\sigma_r(\tau) - \sigma(\tau)]\,\frac{\partial C(t,\tau)}{\partial \tau}\,d\tau$$

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* For the sake of brevity the arguments r, θ, z are omitted.

$$\varepsilon_\theta(t)\,\varphi[\varepsilon_i(t)] = [\sigma_\theta(t) - \sigma(t)] - \int_{\tau_1}^{t} [\sigma_\theta(\tau) - \sigma(\tau)] \frac{\partial C(t,\tau)}{\partial \tau} d\tau$$

$$\varepsilon_z(t) = 0 \tag{1.3}$$

$$\gamma_{r\theta}(t)\,\varphi[\varepsilon_i(t)] = \tau_{r\theta}(t) - \int_{\tau_1}^{t} \tau_{r\theta}(\tau) \frac{\partial C(t,\tau)}{\partial \tau} d\tau$$

where

$$\sigma(t) = \sigma_z(t) = \tfrac{1}{2}[\sigma_r(t) + \sigma_\theta(t)] \tag{1.4}$$

We note that equations (1.3) describe processes of deformation, taking both aging and heredity of the material into account, and are valid for active deformations; the criterion of activity is the condition of increase of $\epsilon_i(t)$.

These equations are deduced assuming incompressibility of the material:

$$\varepsilon(t) = \varepsilon_r(t) + \varepsilon_\theta(t) = 0 \tag{1.5}$$

Experimental creep curves for metals are sufficiently well described by the power law of the form

$$\varphi^*[\varepsilon_i(t)] = \varphi[\varepsilon_i(t)]\,\varepsilon_i(t) = K_0 \varepsilon_i^{\mu}(t) \tag{1.6}$$

Here K_0 and μ are some physical constants determined from a simple creep test.

The available experimental data regarding the creep of metals and other structural materials under constant loading [7] show that with increase of loading the total deformation usually increases more rapidly than predicted by a linear law. Analytically this condition is expressed by the inequality

$$\frac{\partial^2 \varepsilon}{\partial \sigma^2} > 0 \quad \text{for } t = t_1 = \text{const} \tag{1.7}$$

Therefore, for the creep law (1.1) and (1.6) adopted here, this condition will be satisfied for

$$K_0 > 0, \quad \mu < 1 \tag{1.8}$$

2. Equilibrium of the half-plane, subjected to a concentrated force, applied to its free surface, under the conditions of nonlinear creep. Let us consider the quasi-static problem of equilibrium of a half-plane subjected to a time-dependent concentrated force $P(t)$ applied to its free surface, taking creep of the material into account, and for a power law (1.3) and (1.6) relating stresses and strains.

This problem, for plasticity with a power-dependent hardening in terms of stresses, was solved by Sokolovskii [8], who determined the stress and strain distribution in the half-plane under the conditions of simultaneous action of a vertical and a horizontal force applied to its surface.

However, the problem of determining the displacements in the half-plane considered, using the given components of strain presented in paper [8], is reduced to the solution of differential equations with variable coefficients, which are not integrable in closed form.

In this section, based on the fundamental equations (1.3) and (1.6) of the nonlinear theory of creep, the solution of this problem directly in terms of displacements is given, because it is precisely in this form that we will need this solution subsequently, when studying the plane contact problem of the theory of creep.

We place the origin of a cylindrical system of coordinates $(r,\ \theta,\ z)$ at the point of application of the concentrated force $P(t)$ to the half-plane and direct the axes r, θ and x, as indicated in Fig. 1.

Then, solving equations (1.3) which relate the components of strain with the components of stress under the conditions of nonlinear creep, with respect to $[\sigma_r(t) - \sigma(t)]$, $[\sigma_\theta(t) - \sigma(t)] + \tau_{r\theta}(t)$ and taking into account relations (1.4), (1.5) and (1.6) we obtain

$$
\begin{gathered}
\sigma_r(t) = \sigma_\theta(t) + 2K_0\left\{\varepsilon_r(t)\,\varepsilon_i^{\mu-1}(t) + \int_{\tau_1}^{t} \varepsilon_r(\tau)\,\varepsilon_i^{\mu-1}(\tau)\,R(t,\tau)\,d\tau\right\} \\
\tau_{r\theta}(t) = K_0\left\{\gamma_{r\theta}(t)\,\varepsilon_i^{\mu-1}(t) + \int_{\tau_1}^{t} \gamma_{r\theta}(\tau)\,\varepsilon_i^{\mu-1}(\tau)\,R(t,\tau)\,d\tau\right\} \qquad (2.1) \\
\sigma_z(t) = \tfrac{1}{2}\left[\sigma_r(t) + \sigma_\theta(t)\right]
\end{gathered}
$$

where $R(t, \tau)$ is the resolvent of the creep kernel $K(t, \tau) = \partial C(t, \tau)/\partial\tau$, that is, the relaxation kernel.

The equations of equilibrium in the cylindrical system of coordinates $(r,\ \theta,\ z)$ applicable in the given problem are of the form:

$$
\frac{\partial}{\partial r}\left[r\sigma_r(t)\right] + \frac{\partial \tau_{r\theta}(t)}{\partial\theta} - \sigma_\theta(t) = 0, \qquad \frac{\partial\sigma_\theta(t)}{\partial\theta} + r\frac{\partial\tau_{r\theta}(t)}{\partial r} + 2\tau_{r\theta}(t) = 0 \quad (2.2)
$$

The relations between the components of strain and the components of displacement are

$$\varepsilon_r(t) = \frac{\partial u(t)}{\partial r}, \qquad \varepsilon_\theta = \frac{1}{r}\frac{\partial v(t)}{\partial \theta} + \frac{u(t)}{r}, \qquad \varepsilon_z(t) = \frac{\partial w(t)}{\partial z} = 0$$

$$2\gamma_{r\theta}(t) = \frac{\partial v(t)}{\partial r} - \frac{v(t)}{r} + \frac{1}{r}\frac{\partial u(t)}{\partial \theta} \tag{2.3}$$

where $u(t)$, $v(t)$, the components of displacement along the directions of coordinates (r, θ, z) at the instant of time t, yield the differential equation of strain compatibility in the form

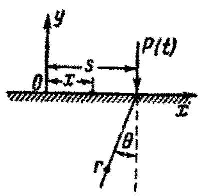
$$\frac{\partial^2 \varepsilon_r(t)}{\partial \theta^2} + r^2 \frac{\partial^2 \varepsilon_\theta(t)}{\partial r^2} + 2r\frac{\partial \varepsilon_\theta(t)}{\partial r} - r\frac{\partial \varepsilon_r(t)}{\partial r} - 2r\frac{\partial^2 \gamma_{r\theta}(t)}{\partial r\, \partial \theta} - 2\frac{\partial \gamma_{r\theta}(t)}{\partial \theta} = 0 \tag{2.4}$$

The boundary conditions of the problem are

$$\sigma_\theta(t) = \tau_{r\theta}(t) = 0 \qquad \text{for } \theta = \pm \tfrac{1}{2}\pi \tag{2.5}$$

that is, the free surface of the half-plane is free of external loading.

We shall seek an exact solution of the formulated problem in terms of displacements in the following form:



Фиг. 1

$$u(t) = \varkappa\,[f_1(r)\,\chi'(\theta, t) + f_0'(\theta, t)]$$
$$v(t) = \varkappa\,[f_2(r)\,\chi(\theta, t) - f_0(\theta, t)] \tag{2.6}$$
$$w = 0, \qquad \varkappa = \pm 1$$

where $f_1(r)$, $f_2(r)$, $\kappa(\theta, t)$ and $f_0(\theta, t)$ are some single-valued and continuous functions, to be defined in the whole half-space $-\pi/2 < \theta < \pi/2$ and $r > 0$ at any instant of time $t \geqslant \tau_1$.

From the first two relations (2.3) we have

$$\varepsilon_r(t) = \varkappa f_1'(r)\,\chi'(\theta, t), \qquad \varepsilon_\theta(t) = \varkappa \frac{1}{r}[f_2(r) + f_1(r)]\,\chi'(\theta, t) \tag{2.7}$$

Using relation (2.7) and the condition of incompressibility of the material (1.5), we find

$$f_2(r) = -[f_1'(r)\,r + f_1(r)] \tag{2.8}$$

We assume that the shear stress $\tau_{r\theta}(t)$ is equal to zero in the whole half-plane for any t. Then by virtue of (2.1) and (2.3) we have

$$\frac{\partial v(t)}{\partial r} - \frac{v(t)}{r} + \frac{1}{r}\frac{\partial u(t)}{\partial \theta} = 0 \tag{2.9}$$

Substituting into (2.9) the expression for the components of displacement and their derivatives from (2.6) and using equation (2.8), we obtain the following equations for the determination of the functions $f_0(\theta, t)$,

$\chi(\theta, t)$ and $f_1(r)$:

$$f_0''(\theta, t) + f_0(\theta, t) = 0, \qquad \chi''(\theta, t) + \lambda^2\chi(\theta, t) = 0 \tag{2.10}$$

$$r^2 f_1''(r) + r f_1'(r) - [1 - \lambda^2] f_1(r) = 0 \tag{2.11}$$

where λ is a parameter to be determined later.

The general integral of the equation (2.11) is

$$f_1(r) = D_1 r^{\sqrt{1-\lambda^2}} + D_2 r^{-\sqrt{1-\lambda^2}} \quad \text{for } -\infty < \lambda^2 < 1 \tag{2.12}$$

Here D_1 and D_2 are the constants of integration.

Admitting the obvious condition that for $r \to \infty$ the displacements $u(t)$ and $v(t)$ should be finite for arbitrary $t \geqslant \tau_1$, we obtain $D_1 = 0$ by virtue of the relations (2.6), (2.8), and (2.12).

Then expression (2.12) for $f_1(r)$ takes on the form

$$f_1(r) = r^{-\sqrt{1-\lambda^2}} \qquad (-\infty < \lambda^2 < 1) \tag{2.13}$$

where for the sake of simplicity of further calculations it is assumed that $D_2 = 1$.

Using relations (2.3), (2.6), (2.7), (2.8), (2.9) and (2.13) we find

$$\varepsilon_r(t) = -\varepsilon_\theta(t) = -\varkappa\sqrt{1-\lambda^2}\, r^{-(1+\sqrt{1-\lambda^2})}\chi'(\theta, t)$$

$$\gamma_{r\theta}(t) = \varepsilon_z(t) = 0 \tag{2.14}$$

The intensity of shear strain $\epsilon_i(t)$, by virtue of relations (1.2) and (2.14) will be

$$\varepsilon_i(t) = |\varepsilon_r(t)| = \sqrt{1-\lambda^2}\, r^{-(1+\sqrt{1-\lambda^2})}\chi'(\theta, t) \tag{2.15}$$

Substituting the expression for the components of strain $\epsilon_r(t)$, $\epsilon_\theta(t)$, $\gamma_{r\theta}(t)$ and $\epsilon_i(t)$ from (2.14) and (2.15) into (2.1), we obtain

$$\begin{gathered}\sigma_r(t) = \sigma_\theta(t) - 2\varkappa K_0[\sqrt{1-\lambda^2}\, r^{-(1+\sqrt{1-\lambda^2})}]^\mu H_1(\theta, t) \\ \sigma_z(t) = \tfrac{1}{2}[\sigma_r(t) + \sigma_\theta(t)], \quad \tau_{r\theta}(t) = 0\end{gathered} \tag{2.16}$$

where

$$H_1(\theta, t) = \xi(\theta, t) + \int_{\tau_1}^{t} \xi(\theta, \tau) R(t, \tau)\, d\tau, \quad \xi(\theta, t) = [\chi'(\theta, t)]^\mu \tag{2.17}$$

$\chi(\theta, t)$ is here the solution of equation (2.10).

The expressions (2.14), (2.15) and (2.16) for the components of strain and stress, by virtue of equation (2.10), identically satisfy both the equations of the theory of plastic heredity (1.3) or (2.1), and also the equations of strain compatibility (2.4).

Substituting expressions for the components of stress from (2.16) into the equilibrium equations (2.2), we find that these equations will be satisfied if we set

$$\sqrt{1-\lambda^2} = \frac{1}{\mu} - 1, \qquad \sigma_\theta = \text{const} \tag{2.18}$$

But on the free surface the stresses vanish, that is

$$\sigma_\theta(t) = \tau_{r\theta}(t) = 0 \qquad \text{for } \theta = \pm\tfrac{1}{2}\pi \text{ and } t \geqslant \tau_1$$

These conditions are compatible with (2.18) only in the case when $\sigma_\theta(t) = 0$ everywhere.

Then expressions (2.16) and (2.18) take on the form:

$$\sigma_r(t) = -\frac{2\varkappa K_0(m-1)^\mu}{r} H_1(\theta, t), \qquad \sigma_\theta(t) = \tau_{r\theta}(t) = 0$$

$$\sigma_z(t) = -\frac{\varkappa K_0(m-1)^\mu}{r} H_1(\theta, t), \qquad \lambda^2 = \frac{2\mu-1}{\mu^2}, \quad m = \frac{1}{\mu} \tag{2.19}$$

where the function $H_1(\theta, t)$ is determined by formula (2.17).

Equating the component of the principal stress vector acting at an arbitrary section of the half-plane bounded by a cylindrical surface r = const, to the vertical force $P(t)$, we obtain an equation which has to be satisfied by the stress $\sigma_r(t)$:

$$P(t) = -\int_{-1/2\pi}^{+1/2\pi} \sigma_r(t)\, r\cos\theta\, d\theta \tag{2.20}$$

From relation (2.19) it follows that for $0 < \mu \leqslant 1$ the parameter λ^2 varies in the range $-\infty < \lambda^2 \leqslant 1$, whereby the sign of the equation $\mu = \lambda^2 = 1$ corresponds in accordance with (2.1) to the case of equilibrium of a half-plane under the conditions of linear creep of the material.

We proceed to the determination of the displacement $u(t)$ and $v(t)$ in the half-plane.

The solution of the first differential equation (2.10) is

$$f_0(\theta, t) = D_5(t)\cos\theta + D_6(t)\sin\theta \tag{2.21}$$

The solution of the second differential equation (2.10) will have a different form depending upon the value of μ. For $\mu = 1/2$ the function $\chi(\theta, t)$ will be linear with respect to θ:

$$\chi(\theta, t) = D_3(t) + D_4(t)\,\theta \tag{2.22}$$

and for $\mu \neq 1/2$ the function $\chi(\theta, t)$ is expressible by means of trigonometric or hyperbolic functions in the following manner:

$$\begin{aligned} \chi(\theta, t) &= D_3(t)\cos\lambda\theta + D_4(t)\sin\lambda\theta \qquad \left(\mu > \tfrac{1}{2}\right) \\ \chi(\theta, t) &= D_3(t)\,\mathrm{ch}\,\lambda\theta + D_4(t)\,\mathrm{sh}\,\lambda\theta \qquad \left(\mu < \tfrac{1}{2}\right) \end{aligned} \tag{2.23}$$

Here $D_3(t)$, $D_4(t)$, $D_5(t)$ and $D_6(t)$ are arbitrary functions of integration which depend only on t.

We assume that the half-plane considered is not being displaced in the horizontal direction and is not rotated, so that

$$v(t) = 0 \quad \text{for } \theta = 0 \text{ and } t \geqslant \tau_1 \tag{2.24}$$

Then in accordance with (2.6), (2.21), (2.22) and (2.23) we will have

$$D_3(t) = D_5(t) = 0 \tag{2.25}$$

and the expressions (2.21), (2.22) and (2.23) for functions $f_0(\theta, t)$ and $\chi(\theta, t)$ take on the form

$$\begin{aligned} f_0(\theta, t) &= D_6(t)\sin\theta \\ \chi(\theta, t) &= D_4(t)\,\theta \qquad \left(\mu = \tfrac{1}{2}\right) \\ \chi(\theta, t) &= D_4(t)\sin l\theta \qquad \left(\mu > \tfrac{1}{2}\right) \\ \chi(\theta, t) &= D_4(t)\,\mathrm{sh}\,\beta\theta \qquad \left(\mu < \tfrac{1}{2}\right) \end{aligned} \tag{2.26}$$

where l and β are related to μ as

$$l^2 = \frac{2\mu - 1}{\mu^2}, \qquad \beta^2 = \frac{1 - 2\mu}{\mu^2} \tag{2.27}$$

Now, using equations (2.17), (2.19), (2.20) and (2.26), for the determination of $D_4(t)$, we obtain the following Volterra integral equation

$$D_1^*(t) = \frac{P(t)}{K_0 (m-1)^{\mu} J(\mu)} - \int_{\tau_1}^{t} D_4^*(\tau) R(t, \tau)\, d\tau \tag{2.28}$$

Here

$$D_4^*(t) = D_4^{\mu}(t), \qquad m = \frac{1}{\mu}, \qquad \varkappa = +1$$

$$J(\mu) = 4l^{\mu} \int_0^{1/2\pi} (\cos l\theta)^{\mu} \cos\theta\, d\theta \qquad \left(\mu > \tfrac{1}{2}\right),$$

$$J(\mu) = 4\beta^{\mu} \int_0^{1/2\pi} (\operatorname{ch} \beta\theta)^{\mu} \cos\theta \, d\theta \qquad \left(\mu < \tfrac{1}{2}\right), \quad J(\mu) = 4 \qquad \left(\mu = \tfrac{1}{2}\right) \quad (2.29)$$

K_0 and μ are the physical constants which characterize the power law of the nonlinearity (1.6).

The solution of equations (2.28) is

$$D_4(t) = \frac{1}{K_0^m (m-1) J^m(\mu)} \left(P(t) - \int_{\tau_1}^{t} P(\tau) \frac{\partial C(t, \tau)}{\partial \tau} \, d\tau\right)^m \qquad (2.30)$$

since $R(t, \tau)$ is the resolvent of the creep kernel $K(t, \tau) = \partial C(t, \tau)/\partial\tau$.

Substituting into relations (2.6) the expressions of the functions $\chi(\theta, t)$, $f_0(\theta, t)$, $f_1(r)$ and $f_2(r)$ and their derivatives from (2.26), (2.13), (2.8) and using the expression for the function $D_4(t)$ (2.30), we obtain

$$\begin{aligned} u(t) &= \frac{[(1-\mathbf{L}) P(t)]^m}{K_0^m (m-1) J^m(\mu)} r^{1-m} \eta'(\theta, \mu) + D_6(t) \cos\theta \\ v(t) &= \frac{(m-2)[(1-\mathbf{L}) P(t)]^m}{K_0^m (m-1) J^m(\mu)} r^{1-m} \eta(\theta, \mu) - D_6(t) \sin\theta \end{aligned} \qquad (2.31)$$

Here $\mathbf{L}$ is the Volterra integral operator of the form

$$Ly(t) = \int_{\tau_1}^{t} y(\tau) \frac{\partial C(t, \tau)}{\partial \tau} \, d\tau \qquad (2.32)$$

$$\eta(\theta, \mu) = \theta \quad \left(\mu = \tfrac{1}{2}\right), \quad \eta(\theta, \mu) = \sin l\theta \quad \left(\mu > \tfrac{1}{2}\right) \qquad (2.33)$$

$$\eta(\theta, \mu) = \operatorname{sh} \beta\theta \qquad \left(\mu < \tfrac{1}{2}\right)$$

The displacements of the boundary points of the half-plane, that is for $\theta = \pm \pi/2$, in accordance with (2.31) and (2.33), will be of the form:

$$[u(t)]_{\theta=-1/2\pi} = [u(t)]_{\theta=1/2\pi} = B[(1-\mathbf{L}) P(t)]^m r^{1-m} \qquad (2.34)$$

$$[v(t)]_{\theta=-1/2\pi} = -[v(t)]_{\theta=1/2\pi} = A[(1-\mathbf{L}) P(t)]^m r^{1-m} + D(t)$$

Here

$$\begin{aligned} &A = 0, && B = \frac{1}{16K_0^2} && \left(\mu = \tfrac{1}{2}\right) \\ &A = \frac{(2-m)\sin {}^1\!/_2 l\pi}{K_0^m (m-1) J^m(\mu)}, && B = \frac{l \cos {}^1\!/_2 l\pi}{K_0^m (m-1) J^m(\mu)} && \left(\mu > \tfrac{1}{2}\right) \\ &A = \frac{(2-m)\operatorname{sh} {}^1\!/_2 \beta\pi}{K_0^m (m-1) J^m(\mu)}, && B = \frac{\beta \operatorname{ch} {}^1\!/_2 \beta\pi}{K_0^m (m-1) J^m(\mu)} && \left(\mu < \tfrac{1}{2}\right) \end{aligned} \qquad (2.35)$$

The equations for $u(t)$ and $v(t)$ obtained above are valid when the material is under the conditions of nonlinear creep, that is, for $0 < \mu < 1$.

We note that, as follows from formulas (2.34) and (2.35), for a quadratic law of nonlinearity, that is, for $\mu = 1/2$ and $m = 2$, all boundary points of the half-plane undergo instantaneous rigid displacements in the vertical direction which are equal to

$$v(t) = -\boldsymbol{v}(t) = D(t) \text{ for } \theta = \pm \frac{\pi}{2}$$

Using relations (2.19) and noting that in accordance with (2.17), (2.26) and (2.28)

$$\xi(\theta, t) = D_4^*(t)\,[\eta'(\theta, \mu)]^{\mu} = D_4^{\mu}(t)\,[\eta'(\theta, \mu)]^{\mu}$$

$$H_1(\theta, t) = \frac{P(t)\,[\eta'(\theta, \mu)]^{\mu}}{K_0\,(m-1)^{\mu}\,J(\mu)} \tag{2.36}$$

we obtain the following final formulas for the stresses $\sigma_r(t)$ and $\sigma_z(t)$:

$$\sigma_r(t) = -\frac{2P(t)\,[\eta'(\theta, \mu)]^{\mu}}{rJ(\mu)}, \qquad \sigma_z(t) = -\frac{P(t)\,[\eta'(\theta, \mu)]^{\mu}}{rJ(\mu)} \tag{2.37}$$

$$\sigma_{\theta}(t) = \tau_{r\theta}(t) = 0$$

which, for each fixed instant of time $t = t_1$, coincide with the formulas for the stress in the half-plane under the conditions of plasticity, with a power law governing the hardening of the material given in paper [9]. These formulas were obtained in a different manner by Sokolovskii [9].

Thus, the stress distribution in the half-plane considered (2.37), coincides identically with the system of stresses which correspond to an instantaneous problem of the nonlinear theory of elasticity for the same half-plane, even though the strain rates turn out not to be constant here, but are variable, since the factor $D_{\mu}(t)$ to be determined from Volterra's integral equation (2.28) depends on time t. Under the conditions of nonlinear creep this is explained by the fact that the system of equations (1.3) and (1.6), even though it represents the equations of steady creep, understood in a broader sense than the usual one, is still reduceable, by means of the elastic analogy as in the usual case, to the corresponding instantaneous problems of the nonlinear theory of elasticity.

3. The plane contact problem of the theory of creep. *1. Formulation of problem and deduction of basic equations.* Using the nonlinear - elastic analogy as a basis, we consider in a general form the solution of the contact problem of two bodies bounded by smooth surfaces which are in the conditions of nonlinear creep, obeying the power law (1.6) which relates the deformations with stresses.

Let two bodies, in contact with each other at a point or along a line

(Fig. 2), and which possess the property of creep, be compressed by means of external forces whose resultant P is perpendicular to the x-axis and which passes through the origin of the system of coordinates.

The relation, which has to be satisfied by the displacements of the points of the contact region of these bodies, is of the form:

$$v_1(t) + v_2(t) = \delta(t) - f_1^*(x) - f_2^*(x) \tag{3.1}$$

where $\delta(t) = \delta_1(t) + \delta_2(t)$ is the approach of these bodies in the direction of the axis oy, and $f_1^*(x)$ and $f_2^*(x)$ are the equations of the surfaces, bounding the first and the second bodies.

We shall assume further that friction and cohesion between the two compressed bodies are absent. Then, on the contact area each of these bodies will experience only a normal pressure which will be designated by $p(x, t)$. However, the contact region will usually be small as compared to the dimensions of the compressed bodies, and therefore, it may be assumed that the displacements on the contact area of the compressed bodies will be the same as those of the boundary points of two half-planes (an upper and a lower), subjected to the action of the same normal pressure $p(x, t)$ as the compressed bodies considered.

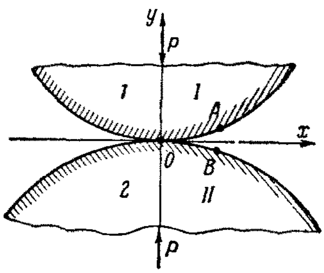


Fig. 2.

We divide the pressure diagram $p(x, t)$, acting on the contact area $S(a \leqslant x \leqslant b)$, into elementary strips of widths Δs_i and height $p(s_i, t)$ $(i = 1, \ldots, n)$ and consider the effect of one such strip (for example, ith) on the lower half-plane.

If a concentrated force $P_i(t) = p(s_i, i)\Delta s_i$ is applied at the point $x = s_i$ normal to the boundary of the half-plane, then the boundary point of this half-plane with abscissa x will be displaced by $v(t)$ in the direction of the axis oy, which can be determined in accordance with formula (2.34)

$$v(t) = A[(1 - L)P_i(t)]^m |s_i - x|^{1-m} + D(t) \tag{3.2}$$

or in a different form:

$$v^*(t) = h_i(t)\, p(s_i, t)\, \Delta s_i \tag{3.3}$$

where

$$h_i(t) = A^\mu |s_i - x|^{\mu-1}(1 - L), \qquad v^*(t) = [v(t) - D(t)]^\mu$$

$$m = \frac{1}{\mu}, \qquad LP_i(t) = \int_{\tau_1}^{t} P_i(\tau) \frac{\partial C(t, \tau)}{\partial \tau} d\tau \tag{3.4}$$

In what follows, $v^*(x, t)$ will be called the generalized displacement of the boundary points of the half-plane.

For the case of simultaneous action of a system of forces $P_i(t) = p(s_i, t)\Delta s_i (i = 1, \ldots, n)$ the generalized displacement $v^*(x, t)$ of the arbitrary point on the boundary of the half-plane, will be in general a certain function of these forces $v^*(x, t) = v^*[P_1(t)\ P_2(t), \ldots, P_n(t)]$, which can be represented in the form

$$v^*(t) = \sum_{j=1}^{j=n} C_j(t)\, p(s\ , t)\, \Delta s_j + \sum_{k=1}^{k=n} \sum_{j=1}^{j=n} C_{jk}(t)\, p(s_k, t)\, p(s_j t)\, \Delta s_k\, \Delta s_j + $$
$$+ \sum_{\nu=1}^{\nu=n} \sum_{k=1}^{k=n} \sum_{j=1}^{j=n} C_{\nu kj}(t)\, p(s_\nu, t)\, p(s_k, t)\, p(s_j, t)\, \Delta s_\nu\, \Delta s_k\, \Delta s_j + \ldots \tag{3.5}$$

where $C_j(t)$, $C_{jk}(t)$ and $C_{\nu jk}(t)$ are certain coefficients which depend also on x and $s_i (i = 1, \ldots, n)$, which are omitted for the sake of brevity in the notation.

However, on the other hand, if only one force is acting, that is, if $P_i(t) = 0$ for $j \neq i$ and $P_j(t) = P_i(t)$ for $j = i$, expression (3.5) for $v^*(t)$ should coincide identically with the exact solution of this problem, given by formula (3.3). As a consequence we will have

$$C_i(t) = h_i(t), \qquad C_{ii}(t) = 0, \qquad C_{iii}(t) = 0 \tag{3.6}$$

and the expression for the generalized displacement $v^*(t)$ takes on the form:

$$v^*(t) = \sum_{j=1}^{j=n} h_j(t)\, p(s_j, t)\, \Delta s_j +$$

$$+ \sum_{j \neq k}^{n} C_{jk}(t)\, p(s_j, t)\, p(s_k, t)\, \Delta s_j\, \Delta s_k + \ldots \quad (j, k = 1, \ldots, n) \tag{3.7}$$

Since the area of contact $S(a < x \leqslant b)$ is small, it.is possible to retain in expression (3.7) for the generalized displacement $v^*(t)$ only the principal term of the expansion, staying within the same range of accuracy as was assumed in solving the given problem. We then obtain from expression (3.7), after passing to the limit as $\Delta s_i \to 0$

$$v^*(t) = A^\mu [1 - L] \int_S \frac{p(s, t)\, ds}{|s - x|^{1-\mu}} \tag{3.8}$$

where the integration is carried over the whole area of contact $S(a \leqslant x \leqslant b)$, which in the general case will be dependent on time; hence, using formula (3.4) which determines the operator L, we obtain the generalized displacement $v^*(t)$. For the displacement $v(t)$ of the contact points, we obtain

$$v(t) = A\left[(1-L)\int_S \frac{p(s,t)\,ds}{|s-x|^{1-\mu}}\right]^m + D(t)$$

where $m = 1/\mu$ and the constant A is determined in accordance with (2.35).

If the same normal pressure $p(x, t)$ were acting on the boundary of the upper half-plane, then the boundary point with abscissa x would undergo a displacement oy in the direction of the axis $v(t)$, which will be equal to

$$v(t) = -A\left[(1-L)\int_S \frac{p(s,t)\,ds}{|s-x|^{1-\mu}}\right]^m + D(t) \tag{3.9}$$

Thus, under the assumptions stated above, the expressions for the displacements $v_1(t)$ and $v_2(t)$ for the first and second body, in accordance with (3.8) and (3.9), will be

$$v_1(t) = A_1\left[(1-L)\int_S \frac{p(s,t)\,ds}{|s-x|^{1-\mu}}\right]^m + D_1(t)$$

$$v_2(t) = A_2\left[(1-L)\int_S \frac{p(s,t)\,ds}{|s-x|^{1-\mu}}\right]^m + D_2(t) \tag{3.10}$$

Here (3.11)

$$A_1 = \frac{(2-m)\sin {}^1\!/_2\, l\pi}{K_{01}{}^m (m-1) J^m(\mu)}, \qquad A_2 = \frac{(2-m)\sin {}^1\!/_2\, l\pi}{K_{02}{}^m (m-1) J^m(\mu)} \qquad (\mu > \tfrac{1}{2})$$

$$A_1 = \frac{(2-m)\,\mathrm{sh}\, {}^1\!/_2\, \beta\pi}{K_{01}{}^m (m-1) J^m(\mu)}, \qquad A_2 = \frac{(2-m)\,\mathrm{sh}\, {}^1\!/_2\, \beta\pi}{K_{02}{}^m (m-1) J^m(\mu)} \qquad (\mu < \tfrac{1}{2})$$

$$A_1 = A_2 = 0 \qquad (\mu = \tfrac{1}{2})$$

where K_{01} and K_{02} are physical constants of the first and the second body.

Substituting the expression for $v_1(t)$ and $v_2(t)$ from (3.10) into the relation (3.1) for the determination of pressure $p(x, t)$ we obtain the following integral equation:

$$\int_S \frac{p(s,t)\,ds}{|s-x|^{1-\mu}} - \int_{\tau_1}^{t}\int_S \frac{p(s,\tau)\,ds}{|s-x|^{1-\mu}}\,\frac{\partial C(t,\tau)}{\partial\tau}\,d\tau = [\gamma(t) - f_0(x)]^\mu \tag{3.12}$$

where

$$f_0(x) = \frac{f_1^*(x) + f_2^*(x)}{A_1 + A_2}$$

and where $f_0(x)$ does not depend on t, $S(a \leqslant x \leqslant b)$ is the contact width, which will in general be a function of time, and $\gamma(t)$ is the unknown function of t, to be determined later.

The integral equation (3.12) may be represented in a more compact form, namely, as the following two integral equations:

$$\omega(x,t) - \int_{\tau_1}^{t} \omega(x,\tau) \frac{\partial C(t,\tau)}{\partial \tau} d\tau = [\gamma(t) - f_0(x)]^{\mu} \tag{3.13}$$

$$\int_S \frac{p(s,t)\, ds}{|s - x|^{1-\mu}} = \omega(x,t) \tag{3.14}$$

Here and subsequently , for the sake of brevity, we shall designate by $\omega(x, t)$ the function which, being the solution of the integral equation (3.13), depends both on the arguments x and t, as well as on the unknown function $\gamma = \gamma(t)$, which enters into the right-hand side of this equation, that is $\omega(x, t) = \omega^*[x, t, \gamma(t)]$.

Thus, the solution of the plane contact problem of the nonlinear theory of creep which in essence consists in finding the unknown function of the two variables $p(x, t)$, which characterize the distribution of the pressure intensity along the contact of compressed bodies, is reduced to a simultaneous solution of two coupled integral equations (3.13) and (3.14).

The first of these, which has to be satisfied by $\omega(x, t)$ as a function of time t, takes into account the effect of creep of the material on the distribution of contact forces, and represents a linear Volterra integral equation of the second kind, which, for various cases of creep kernels $K(t, \tau) = \partial C(t, \tau)/\partial\tau$ was investigated in detail in the publications [2, 3, 10].

The second integral equation (3.14), which has to be satisfied by $p(x, t)$, as a function of the argument x, represents a singular Fredholm integral equation of the first kind with kernel

$$K(s,x) = |s - x|^{\mu - 1} \qquad (0 < \mu < 1)$$

and with the right-hand side $\omega(x, t)$, which is the solution of the first integral equation (3.13) and which may be considered at each fixed t as a basic integral equation of some plane contact problem of the nonlinear theory of elasticity, whose method of solution is presented further below in subsection 2-4 of the present section. We shall note that for $t = \tau_1$ we obtain from the general solution of the basic equations of the plane contact problem of the nonlinear theory of creep (3.13) and (3.14) directly the solution of the contact problem of the theory of plasticity with a hardening of the material obeying a power law, as presented in paper [9].

In fact, for $t = \tau_1$ we have in accordance with (3.13)

$$\omega(x, \tau_1) = \omega(x) = [\gamma - f_0(x)]^{\mu}$$

and the integral equation (3.14) takes on the form

$$\int_S \frac{p(s)\,ds}{|s-x|^{1-\mu}} = [\gamma - f_0(x)]^{\mu} \tag{3.15}$$

which, as is shown in paper [9], is the basic integral equation of the plane contact problem of the theory of plasticity for a material with a strain hardening which follows a power law.

If $C(t, \tau) \equiv 0$ and $\gamma(t) = \gamma = \text{const}$, that is, if the material of the compressed bodies does not exhibit creep, then we again are led to the contact problem of the nonlinear theory of elasticity, described by equation (3.15).

2. *Solution of the basic integral equation (3.14) of the plane contact problem of the nonlinear theory of creep.* Let the initial contact of the bodies to be compressed in the plane xy be at a point, which is taken as the origin of the coordinate system (Fig. 2).

We assume further that the portion of the axis ox, $-a(t) \leqslant x \leqslant +a(t)$. is the contact region S between these bodies, which in general will be time-dependent.

Then the basic integral equation (3.14) of the plane contact problem takes on the form:

$$\int_{-a(t)}^{+a(t)} \frac{p(s,t)\,ds}{|s-x|^{1-\mu}} = \omega(x, t) \tag{3.16}$$

where $\omega(x, t)$ is the solution of the Volterra integral equation (3.13) which is investigated in detail for various cases of creep kernels $K(t, \tau) = \partial C(t, \tau)/\partial\tau$ in the publications [2,3,10]; therefore we will not dwell upon this item, assuming in what follows, that $\omega(x, t)$ is known, or may be found by using methods developed in these works.

As was already indicated, $\omega(x, t)$, being a continuous function in the region $a(t) \leqslant x \leqslant a(t)$ and $\tau_1 \leqslant t \leqslant \infty$, depends also on the unknown function $\gamma(t)$, which enters into the right-hand side of the Volterra integral equation (3.13), that is $\omega(x, t) = \omega^*[x, t, \gamma(t)]$.

The limitations, imposed on $\omega(x, t)$, as well as the equations which determine $\gamma(t)$ will be given in the following.

Equation (3.16) was first studied by Carleman [11]. In a recently published paper by Akhiezer and Shcherbina [12] another method of solution of this equation is given, making use of transformation formulas of singular integrals.

To solve this singular integral equation (3.16) we use in the present paper the method suggested by Krein [13], applicable for the solution of Fredholm integral equations of the first and second kind with kernels of the form

$$K(s, x) = H(|s - x|) \tag{3.17}$$

This method permits us to obtain the solution of such equations in closed form for a series of other kernels of the type (3.17). Furthermore for the known cases the application of this method yields solutions, which differ from the known ones by the fact that they do not contain singular integrals, taken in the sense of Cauchy.

It should be noted that the solution of the equation of the contact problem of the linear theory of elasticity free of singular integrals was first obtained by Rostovtsev [14].

Let us designate by $g(s, a)$ the solution of the equation (3.16) for $\omega(x, t) = 1$. Then the general solution of equation (3.16), in accordance with [13], will be expressed by the formula

$$\begin{aligned} p(x, t) = \frac{1}{2M'(a)}\left[\frac{d}{da}\int_{-a}^{+a} g(s, a)\,\omega(s, t)\,ds\right] g(x, a) - \\ -\frac{1}{2}\int_{x}^{a} g(x, u)\frac{d}{du}\left[\frac{1}{M'(u)}\frac{d}{du}\int_{-u}^{+u} g(s, u)\,\omega(s, t)\,ds\right]du - \\ -\frac{1}{2}\frac{d}{dx}\int_{x}^{a}\frac{g(x, u)\,du}{M'(u)}\int_{-u}^{+u} g(s, u)\,\omega'(s, t)\,ds \end{aligned} \tag{3.18}$$

Here

$$M(u) = \int_{0}^{u} g(s, u)\,ds, \qquad \omega'(s, t) = \frac{\partial\omega(s, t)}{\partial s} \tag{3.19}$$

$2a = 2a(t)$ is the width of the contact which, in general, depends on time t, but t, for the sake of brevity is omitted in (3.18); $\gamma(t)$ is an unknown function of t, which enters into the right-hand side of the integral equation (3.13) and therefore

$$\omega(x, t) = \omega^{*}[x, t, \gamma(t)]$$

If the contact width $2a(t) = 2a$ is given, then $\gamma(t)$ is determined from the equilibrium equation

$$P = \int_{-a}^{+a} p(x, t)\, dx \tag{3.20}$$

where P is the resultant of the external forces acting on the compressed body, whereby in deducing equation (3.16) it was assumed that P is perpendicular to the x-axis and passes through the origin of coordinates.

We now assume that any constraints inhibiting the rotations of compressed bodies are absent. We construct the basic equation of the contact problem under these conditions.

Relation (3.1), which connects the displacement of the boundary points of the compressed bodies $v_1(t)$ and $v_2(t)$, was obtained under the supposition that in compression these bodies undergo only translational displacements $\delta_1(t)$ and $\delta_2(t)$ in the direction of the oy axis and that therefore they approach each other by an amount equal to $\delta(t) = \delta_1(t) + \delta_2(t)$.

Let us now assume that these bodies undergo upon compression not only translational displacements $\delta_1(t)$ and $\delta_2(t)$ along the y-axis, but also a rotation with respect to the origin of coordinates through angles $a_1(t)$ and $a_2(t)$, respectively, whereby the positive sense will be taken to be the counter-clockwise one. Then, an additional approach will occur between the boundary points of the compressed bodies on the abscissa x, equal to $a_0(t)x$, where $a_0(t) = a_1(t) + a_2(t)$. In order to obtain for this case the condition which must be satisfied by the displacements of the contact points $v_1(t)$ and $v_2(t)$ of the compressed bodies, the constant approach $\delta(t)$ in the relation (3.1) should be replaced by the variable approach $\delta_0(t) + a_0(t)x$. We will therefore have

$$v_1(t) + v_2(t) = \delta(t) + \alpha_0(t)\, x - f_1^*(x) - f_2^*(x) \tag{3.21}$$

Substituting into (3.21) the expressions for $v_1(t)$ and $v_2(t)$ from (3.10), we arrive at the same integral equation (3.12), with the only difference that on the right-hand side instead of the function $F(x,\ t,\ \gamma(t)) = [\gamma(t) + f_0(x)]^{\mu}$ there will be

$$F[x, t, \gamma(t), \alpha(t)] = [\gamma(t) + \alpha(t)\, x - f_0(x)]^{\mu} \tag{3.22}$$

where

$$\alpha(t) = \frac{\alpha_0(t)}{A_1 + A_2}, \qquad f_0(x) = \frac{f_1^*(x) + f_2^*(x)}{A_1 + A_2} \tag{3.23}$$

and the solution of the Volterra integral equation (3.23), which as before will be designated by $\omega(x,\ t)$, will contain not one but two unknown functions: $\gamma = \gamma(t)$ and $a = a(t)$, that is, in this case $\omega(x,\ t) = \omega^*[x,,\ t,\ \gamma(t),\ a(t)]$; the values of the function $\gamma(t)$ and $a(t)$ for given contact with $2a$ are determined from the equilibrium equations

$$P = \int_{-a}^{+a} p(x, t)\, dx, \qquad M_0 = \int_{-a}^{a} p(x, t)\, x\, dx \tag{3.24}$$

where P is the sum of the projections on the y-axis of all external forces acting on the compressed body and M is the moment of these same forces with respect to the origin of coordinates.

We note that, as follows from the paper [13], formula (3.18) supplies the unique integrable solution of equation (3.14), if $M'(a) \neq 0 (0 < a < b)$, where b is some finite constant and the function $\omega(x, t)$ is differentiable and such that upon its substitution into formula (3.18) the integrals which contain this function would be meaningful.

We proceed to the determination of the function $g(s, a)$, that is to the solution of the singular integral equation

$$\int_{-a}^{+a} \frac{g(s, a)\, ds}{|s - x|^{1-\mu}} = 1 \tag{3.25}$$

To this end, and following the idea of Krein [13], we consider the integral

$$I_0 = \frac{1}{2\pi i} \int_{\Gamma} f(z)\, dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{(z^2 - a^2)^{\frac{1}{2}\mu} (z - x)^{1-\mu}} \tag{3.26}$$

taken along the contour, composed of an external circle Γ_R of radius R and the inner contour $ABCDEFKLGNMQA$ which is designated by Γ_a (Fig.3).

First of all it is not difficult to convince oneself that the integral function

$$f^\circ(z) = \frac{1}{f(z)} = (z^2 - a^2)^{\frac{1}{2}\mu} (z - x)^{1-\mu} \qquad (0 < \mu < 1)$$

is divided on the external portion $(-a, a)$ into three equal branches. In fact, we set $\phi_1 = \arg(z + a)$, $\phi_2 = \arg(z - a)$ and $\phi_3 = \arg(z - x)$. In passing in the counter clock-wise sense along the arbitrary closed contour Γ_0, indicated by a broken line in Fig. 3, ϕ_1, ϕ_2 and ϕ_3 will undergo increments of 2π, and therefore $\arg f(z) = 1/2(\phi_1 + \phi_2)\mu + \phi_3(1 - \mu)$ will undergo an increment 2π and $f(z)$ will return to the initial value.

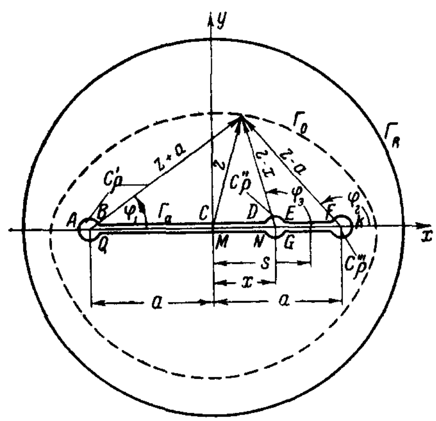


Fig. 3.

We shall consider that branch of the functions $f(z)$, which at the upper boundary $(-a, a)$ will take on a positive value, that is

$$(z-x)^{1-\mu} > 0, \quad (z^2-a^2)^{1/2\mu} > 0 \quad \text{for } z > 0$$

Then, in accordance with Cauchy's theorem for multiply-connected regions we have

$$\frac{1}{2\pi i}\int_{\Gamma_R} f(z)\,dz + \frac{1}{2\pi i}\int_{\Gamma_a} f(z)\,dz = 0 \tag{3.27}$$

where

$$f(z) = (z-a)^{-1/2\mu}(z+a)^{-1/2\mu}(z-x)^{\mu-1} \tag{3.28}$$

However, the integrals along the small circles C_ρ', C_ρ'' and C_ρ''' obviously approach zero if $\rho \to 0$ and therefore

$$\frac{1}{2\pi i}\int_{\Gamma_R} f(z)\,dz = \frac{1}{2\pi i}\left[\int_{+a}^{-a} f(s+i0)\,ds + \int_{-a}^{+a} f(s-i0)\,ds\right] \tag{3.29}$$

Here $f(s + i0)$ and $f(s - i0)$ are the values of the functions $f(z)$ at the upper and lower boundaries of the portion $(-a, a)$.

However, noting that $f(s - i0) = \overline{f(s + i0)}$ (where the bar indicates the conjugate function), and changing the sense of integration in the second integral of the relation (3.29), we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_R} f(z)\,dz = -\frac{1}{\pi}\int_{-a}^{+a} \operatorname{Im} f(s+i0)\,ds \tag{3.30}$$

We evaluate the contour integral

$$I_1 = \int_{\Gamma_R} f(z)\,dz \tag{3.31}$$

where $f(z)$ is expressed by formula (3.28). To this end we employ the expansion of our branch $f(z)$ in the neighborhood of the infinite point.

In accordance with (3.28) we have

$$f(z) = \frac{1}{ze^{i\pi}}\left(1-\frac{a^2}{z^2}\right)^{-1/2\mu}\left(1-\frac{x}{z}\right)^{\mu-1} \tag{3.32}$$

where $(1 - a^2/z^2)^{-\mu/2}$ and $(1 - x/z)^{\mu-1}$ indicate those branches of these functions which are positive on the portion (a, ∞) of the x-axis. Expanding the latter by the binomial formula, we find the residue of the selected branch $f(z)$ at the infinite point. It will be equal to $e^{-i\pi}$ (which is the coefficient of $1/z$ with a reversed sign). Then on the basis of the theorem of residues we obtain

$$\int_{\Gamma_R} f(z)\,dz = -2\pi i\, e^{-i\pi} = 2\pi i \tag{3.33}$$

Substituting the value of this integral into relation (3.30), we find

$$\frac{1}{\pi}\int_{-a}^{+a} \operatorname{Im} f(s+i0)\,ds = -1 \tag{3.34}$$

Further, in accordance with (3.28) and Fig. 3, we have

$$\begin{aligned} f(s+i0) &= \exp\left(-\frac{i\pi\mu}{2}\right)\frac{(a^2-s^2)^{-1/2\mu}}{(s-x)^{1-\mu}} && \text{for } x < s+i0 < a \\ f(s+i0) &= \exp\left(-\frac{i\pi\mu}{2}\right)\frac{(a^2-s^2)^{-1/2\mu}}{e^{i\pi(1-\mu)}(x-s)^{1-\mu}} && \text{for } -a < s+i0 < x \end{aligned} \tag{3.35}$$

Substituting the expression for $f(s + i0)$ from (3.35) in (3.34), we finally obtain after transformation

$$\int_{-a}^{+a} \sin\frac{\pi\mu}{2}\,\frac{(a^2-s^2)^{-1/2\mu}}{\pi\,|s-x|^{1-\mu}}\,ds = 1 \tag{3.36}$$

From here it follows directly that

$$g(s,a) = \frac{1}{\pi}\sin\frac{\pi\mu}{2}(a^2 - s^2)^{-1/2\mu} = \frac{\sin {}^1/_2\,\pi\mu}{\pi\sqrt{(a^2-s^2)^{\mu}}} \tag{3.37}$$

which is the solution of the integral equation (3.25).

Using formulas (3.19) and (3.37), we obtain for $M(s)$

$$M(s) = \frac{2\sqrt{\pi}\, s^{1-\mu}}{(1-\mu)\,\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)} \tag{3.38}$$

where $\Gamma(z)$ is the gamma function.

In studying the state of stress in compressed bodies under the conditions of nonlinear creep subsequently, we will consider separately the case of symmetric and anti-symmetric loading of these bodies. This will, first, make the formulas obtained more lucid, and secondly, be appropriate, since each of these loadings represents a significance of its own, corresponding to a definite characteristic deformation of these bodies. It should be noted that the case of arbitrary loading of compressed bodies cannot be obtained, as it follows from (3.18), (3.13), and (3.22), by means of superposition of the two cases indicated above, and must be solved separately as a distinct problem, with the aid of the general formulas (3.18), (3.13), (3.22) and (3.24).

3. *The symmetric contact problem of two bodies under the conditions of nonlinear creep.* Let both the surfaces which bound the compressed bodies, as well as the external forces which act upon them, be symmetrical with respect to the *oy*-axis. Then the equations of these surfaces

$y = f_1{}^*(x)$ and $y = -f_2{}^*(x)$ will be even functions of x, on the strength of which the right-hand side of the basic integral equation (3.12) of the contact problem $F(x, t, \gamma)$ will also be an even function; due to the evenness of functions $g(x, a)$ and $\omega(x, t)$, the last term of the right-hand side of formula (3.18) vanishes and it takes on the form:

$$p(x,t) = \frac{1}{M'(a)}\left[\frac{d}{da}\int_0^a g(s,a)\,\omega(s,t)\,ds\right] g(x,a) - \tag{3.39}$$

$$-\int_x^a g(x,u)\frac{d}{du}\left[\frac{1}{M'(u)}\frac{d}{du}\int_0^u g(s,u)\,\omega(s,t)\,ds\right]du \quad \left(M(a) = \int_0^a g(s,a)\,ds\right)$$

We note that in calculations the second integral in (3.39) is sometimes conveniently represented in the transformed form on the basis of formula [13]

$$\frac{d}{ds}\left[\frac{1}{M'(s)}\frac{dI}{ds}\right] = \frac{1}{M(s)}\frac{d}{ds}\left[\frac{M^2(s)}{M'(s)}\frac{d}{ds}\left(\frac{I}{M(s)}\right)\right] \tag{3.40}$$

Substituting the expressions for $g(s, a)$ and $M(s)$ from (3.37) and (3.38) in (3.39) and using the equation (3.40), we obtain after transformation

$$p(x,t) = K(\mu)\left\{\frac{a^\mu\,\Phi_1'(a,t,\gamma)}{\sqrt{(a^2-x^2)^\mu}} - \int_x^a \frac{du}{\sqrt{(u^2-x^2)^\mu}}\frac{d}{du}[u^\mu\,\Phi_1'(u,t,\gamma)]\right\} \tag{3.41}$$

Here

$$\Phi_1(u,t,\gamma) = \int_0^u \frac{\omega(s,t)\,ds}{\sqrt{(u^2-s^2)^\mu}}, \quad \Phi_1'(u,t,\gamma) = \frac{d}{du}\int_0^u \frac{\omega(s,t)\,ds}{\sqrt{(u^2-s^2)^\mu}} \tag{3.42}$$

$$K(\mu) = \frac{\Gamma\left(\frac{3-\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)\left(\sin\frac{\pi\mu}{2}\right)}{(1-\mu)\,\pi^2\sqrt{\pi}} = \frac{\Gamma\left(\frac{1-\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)\left(\sin\frac{\pi\mu}{2}\right)^2}{2\sqrt{\pi}\,\pi^2} \tag{3.43}$$

$2a = 2a(t)$ is the variable contact width, where $-a(t) \leqslant x \leqslant a(t)$, and $\gamma = \gamma(t)$ is an unknown function of t, which enters into the right-hand side of the integral equation (3.13) and which has to be determined later. We recall that $\omega(x, t) = \omega^*[x, t, \gamma(t)]$. By means of substituting $s = u \sin\phi$, the expression for $\Phi_1(u, t, \gamma)$ from (3.42) may be represented in the form of the following integral with constant limits:

$$\Phi_1(u,t,\gamma) = u^{1-\mu}\int_0^{\frac{1}{2}\pi} \omega(u\sin\varphi, t)\cos^{1-\mu}\varphi\,d\varphi \tag{3.44}$$

Assuming the existence of a continuous and bounded derivative $\omega(s, t)$ for $s > 0$, after differentiation under the integral sign (3.44), we obtain

$$u\Phi_1'(u, t, \gamma) = (1-\mu)\Phi_1(u, t, \gamma) + \int_0^u \frac{\omega'(s, t)\, s ds}{\sqrt{(u^2-s^2)^\mu}} \tag{3.45}$$

Integrating the last term by parts and noting that $\omega'(0, t) = 0$, the relation (3.45) is reduced to the form

$$u\Phi_1'(u, t, \gamma) = (1-\mu)\Phi_1(u_1, t, \gamma) + \frac{1}{2-\mu}\int_0^u (u^2-s^2)^{1-1/2\mu}\omega''(s, t)\, ds \tag{3.46}$$

Hence we obtain by differentiation

$$\frac{d}{du}[u^\mu \Phi_1'(u, t, \gamma)] = u^{\mu-1}\frac{d}{du}\int_0^u \frac{\omega'(s, t)\, s ds}{\sqrt{(u^2-s^2)^\mu}} \tag{3.47}$$

Integrating by parts on the right-hand side in equations (3.47) and differentiating subsequently with respect to u, we obtain by virtue of the evenness of $\omega(x, t)$

$$\frac{d}{du}[u^\mu \Phi_1'(u, t, \gamma)] = u^\mu \int_0^a \frac{\omega''(s, t)\, ds}{\sqrt{(u^2-s^2)^\mu}} \tag{3.48}$$

Substituting this expression into (3.41) we obtain for $p(x, t)$ the following final formula

$$p(x, t) = K(\mu)\left\{\frac{a^\mu \Phi_1'(a, t, \gamma)}{\sqrt{(a^2-x^2)^\mu}} - \int_x^a \frac{u^\mu du}{\sqrt{(u^2-x^2)^\mu}}\int_0^u \frac{\omega''(s, t)\, ds}{\sqrt{(u^2-s^2)^\mu}}\right\} \tag{3.49}$$

Here $\omega(x, t)$ is the solution of the Volterra integral equation (3.13), which will be a function of the unknown $\gamma = \gamma(t)$, that is $\omega(x, t) = \omega^*(x, t, \gamma(t)]$, and the contact width $2a$ in the general case will depend on time t.

In formula (3.49) the first term represents the solution with singularities at the points $x = \pm a$ and should be retained only in the case of given contact widths $2a(t) = 2a$; the unknown function $\gamma = \gamma(t)$ is to be determined from the equilibrium equation

$$P = 2\int_0^a p(x, t)\, dx \tag{3.50}$$

The second term of this formula represents the continuous part of this solution. Substituting the expression for $p(x, t)$ from (3.49) into the equilibrium equation (3.50), we obtain

$$P = 2K(\mu)\left\{a\Phi_1'(a, t, \gamma)\frac{\sin \frac{1}{2}\pi\mu}{2(1-\mu)K(\mu)\pi} - \int_0^a dx \int_x^a \frac{u^\mu du}{\sqrt{(u^2-s^2)^\mu}}\int_0^u \frac{\omega''(s, t)\, ds}{\sqrt{(u^2-s^2)^\mu}}\right\} \tag{3.51}$$

Here the value of the integral has been used

$$I_2(u)=\int_0^u \frac{ds}{\sqrt{(u^2-s^2)^{\mu}}}=\frac{M(u)\pi}{2\sin{}^1/_2\,\pi\mu}=\frac{\sin{}^1/_2\,\pi\mu\, u^{1-\mu}}{2(1-\mu)K(\mu)\pi} \tag{3.52}$$

Changing the order of integration in the last term of the expression (3.51) and using equations (3.52) and (3.43), we have

$$P=\frac{\sin{}^1/_2\,\pi\mu}{(1-\mu)\pi}\left\{a\Phi_1'(a,t,\gamma)-\int_0^a u\,du\int_0^u \frac{\omega''(s,t)\,ds}{\sqrt{(u^2-s^2)^{\mu}}}\right\} \tag{3.53}$$

or changing the order of integration once more and noting that

$$\int_s^a \frac{u\,du}{\sqrt{(u^2-s^2)^{\mu}}}=\frac{(a^2-s^2)^{1-{}^1/_2\mu}}{(2-\mu)} \tag{3.54}$$

we find

$$P=\frac{\sin{}^1/_2\,\pi\mu}{(1-\mu)\pi}\left\{a\Phi_1'(a,t,\gamma)-\frac{1}{2-\mu}\int_0^a (a^2-s^2)^{1-{}^1/_2\mu}\omega''(s,t)\,ds\right\} \tag{3.55}$$

Using further the relation (3.46), equation (3.55) may be cast finally into the form:

$$\Phi_1(a,t,\gamma)=\frac{P\pi}{\sin{}^1/_2\,\pi\mu}, \qquad \Phi_1(a,t,\gamma)=\int_0^a \frac{\omega(s,t)\,ds}{\sqrt{(a^2-s^2)^{\mu}}} \tag{3.56}$$

where $\omega(x, t)$ is the solution of the Volterra integral equation (3.13) with the right-hand side

$$F(x,t,\gamma)=[\gamma(t)-f_0(x)]^{\mu} \qquad \left(f_0(x)=\frac{f_1^*(x)+f_2^*(x)}{A_1+A_2}\right)$$

while A_1 and A_2 are physical constants to be determined by formulas (3.11).

Thus, in case the contact width $2a(+) = 2a$ is not given, the unknown function $\gamma = \gamma(t)$, entering into formula (3.49) for $p(x, t)$, is determined from equation (3.56). If the contact width $2a(t)$ is not given and the contact takes place along smooth surfaces, then the unknown function $\gamma = \gamma(t)$ is determined from the requirement that in formula (3.49) the first term, which represents the solution with singularities, vanishes, i.e.

$$\Phi_1'(a,t,\gamma)=\frac{d}{da}\int_0^a \frac{\omega(s,t)\,ds}{\sqrt{(a^2-s^2)^{\mu}}}=0 \tag{3.57}$$

Here $\omega(s, t)$ is the solution of equation (3.13), while $2a = 2a(t)$ is the variable contact width. Thus, when the contact width $2a = 2a(t)$ is not given, the function $\gamma = \gamma(t)$ is determined from equation (3.57).

Once the function $\gamma = \gamma(t)$ is determined from equation (3.57), the variable contact width $2a(t)$ may be obtained with the aid of the equilibrium equation (3.50). Substituting the expression for $p(x, t)$ from (3.49) into (3.50), and taking (3.57) into account, we obtain after application of the Dirichlet formula

$$\Phi_1(a, t, \gamma) = \frac{P\pi}{\sin {}^1/_2 \pi\mu} \tag{3.58}$$

where $\Phi_1(a, t, \gamma)$ is determined by formula (3.42).

Hence, equation (3.58) for the determination of the variable contact width $2a(t)$ coincides identically with equation (3.56) for the determination of the function $\gamma = \gamma(t)$, when the contact width $2a(t) = 2a$ is prescribed.

As an application let us consider the contact problem of a rigid die with a rectilinear base, given by the width $2a$ on a half-plane under the conditions of nonlinear creep. In this case

$$f_0(x) = 0, \qquad F(x, t, \gamma) = \gamma^\mu(t) \tag{3.59}$$

Then the solution of the Volterra integral equation (3.13) $\omega(x, t)$ will not depend on x and may be represented in the form

$$\omega(t) = \gamma^\mu(t) + \int_{\tau_1}^{t} \gamma^\mu(\tau) R(t, \tau)\, d\tau \tag{3.60}$$

Substituting the value of $\omega(t)$ from (3.60) into equation (3.56) and using equation (3.52), we obtain for the determination of the function $\gamma = \gamma(t)$

$$\gamma^\mu(t) + \int_{\tau_1}^{t} \gamma^\mu(\tau) R(t, \tau)\, d\tau = \frac{2P\Gamma\left(\frac{3-\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)}{a^{1-\mu}\sqrt{\pi}} \tag{3.61}$$

From equations (3.60) and (3.61) it follows immediately that the solution (3.13) of the equation $\omega(t)$ does not depend on time t either, that is

$$\omega(t) = \omega_0 = \frac{2P\Gamma\left(\frac{3-\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)}{a^{1-\mu}\sqrt{\pi}} \tag{3.62}$$

$$\gamma^\mu(t) = \frac{2P\Gamma\left(\frac{3-\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)}{a^{1-\mu}\sqrt{\pi}}[1 + C(t, \tau_1)], \tag{3.63}$$

where $C(t, \tau)$ is the measure of creep of the material of the half-plane.

Substituting the value $\omega(t) = \omega_0$ from (3.62) into (3.49) and noting that in accordance with (3.45) and (3.56)

$$\Phi_1'(a, t, \gamma) = \frac{1-\mu}{a} \frac{P\pi}{\sin {}^1/_2 \pi\mu} \tag{3.64}$$

We finally obtain the following formula for the determination of pressure $p(x, t)$ on the contact area underneath the die:

$$p(x, t) = p(x) = \frac{\Gamma\left(\frac{3-\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right)\sin\frac{\pi\mu}{2}}{a^{1-\mu}\sqrt{\pi}} \frac{P}{\pi\sqrt{(a^2-x^2)^\mu}} \tag{3.65}$$

From the solution obtained (3.65) it is obvious, that if the contact between the compressed bodies is along a straight line, then the creep of the material of these bodies does not influence the law of stress distribution in the contact region and coincides with the stress value, corresponding to the plane contact problem of the theory of plasticity with a power law for strain hardening [9].

For $\mu = 1$, that is, under the conditions of linear plasticity, formula (3.65) acquires the form

$$p(x, t) = p(x) = \frac{P}{\pi\sqrt{a^2-x^2}} \tag{3.66}$$

and coincides with the well-known solutions [1, 16] of the plane contact problem of the linear theory of creep and the linear theory of elasticity, which in the present case obviously coincide identically.

In conclusion we note that if the contact between the compressed bodies is not along a straight line but along curvilinear surfaces, the creep of the material, as is seen from formulas (3.49) and (3.13), will significantly influence the picture of contact stress distribution.

4. *Antisymmetric contact problem of two bodies under the conditions of nonlinear creep.* For an antisymmetric loading the right-hand side of (3.13)

$$F(x, t, \alpha) = [\alpha(t)x - f_0(x)]^\mu \qquad \left(f_0(x) = \frac{f_1^*(x) + f_2^*(x)}{A_1 + A_2}\right) \tag{3.67}$$

where $a(t)$ is some function of t, to be determined later; $f_0(x)$ is an odd function, and A_1 and A_2 are physical constants to be determined by formulas (3.11) and are odd functions in accordance with (3.22), in the contact region of compressed bodies $-a(t) \leqslant x \leqslant a(t)$ (in this case $\gamma = \gamma(t)$ is equal to zero); therefore, its solution $\omega(x, t)$ will be also an off function and then the first two terms on the right-hand side of formula (3.18) will vanish, and it will take on the following form:

$$p(x,t) = -\frac{d}{dx}\int_x^a \frac{g(x_1 u)}{M'(u)}\,du \int_0^u g(s,u)\,\omega'(s,t)\,ds \tag{3.68}$$

Substituting the expression for $a(s, a)$ and $M(s)$ from (3.37) and (3.38) into (3.68) we obtain

$$p(x,t) = -\frac{K(\mu)}{2-\mu}\frac{d}{dx}\int_x^a \frac{d}{du}\left[(u^2-x^2)^{1-\frac{1}{2}\mu}\right]u^{\mu-1}\Phi_2(u,t,\alpha)\,du \tag{3.69}$$

Here

$$\Phi_2(u,t,\alpha) = \int_0^u \frac{\omega'(s,t)\,ds}{\sqrt{(u^2-s^2)^{\mu}}}, \qquad \Phi_2'(u,t,\alpha) = \frac{d}{du}\int_0^u \frac{\omega'(s,t)\,ds}{\sqrt{(u^2-s^2)^{\mu}}} \tag{3.70}$$

The quantity $K(\mu)$ is determined by equations (3.43) and $\omega^*(x, t) = \omega^*[x,t, a(t)]$ represents a solution of the Volterra integral equation (3.13) with the right-hand side of (3.67).

From relations (3.67) and (3.70) it follows that $p(x, t)$ is an odd function and therefore it is sufficient to determine it in the interval $0 \leqslant x \leqslant a(t)$, since $p(-x, t) = -p(x, t)$.

We note that the function $\omega(x, t)$, being the solution of equation (3.13), depends also on the unknown function $a = a(t)$, which is omitted here for the sake of brevity, i.e. $\omega(x, t) = \omega^*[x, t, a(t)]$.

Integrating by parts the right-hand side of equation (3.69), having differentiated first with respect to x, we obtain

$$p(x,t) = K(\mu)\left\{\frac{a^{\mu-1}x\Phi_2(a,t,\alpha)}{\sqrt{(a^2-x^2)^{\mu}}} + x\int_x^a \frac{u^{\mu-2}\left[(1-\mu)\Phi_2(u,t,\alpha) - u\Phi'_2(u,t,\alpha)\right]du}{\sqrt{(u^2-x^2)^{\mu}}}\right\} \tag{3.71}$$

But by analogy with (3.45) we have in this case

$$u\Phi_2'(u,t,\alpha) = (1-\mu)\Phi_2(u,t,\alpha) + \int_0^u \frac{\omega''(s,t)\,s\,ds}{\sqrt{(u^2-s^2)^{\mu}}} \tag{3.72}$$

Substituting this expression into (3.71), we finally obtain the following formula for $p(x, t)$:

$$p(x,t) = K(\mu)\left\{\frac{a^{\mu-1}x\Phi_2(a,t,\alpha)}{\sqrt{(a^2-x^2)^{\mu}}} - x\int_x^a \frac{u^{\mu-2}du}{\sqrt{(u^2-x^2)^{\mu}}}\int_0^u \frac{\omega''(s,t)\,s\,ds}{\sqrt{(u^2-s^2)^{\mu}}}\right\} \tag{3.73}$$

In formula (3.73) the first term represents a solution with singularity at the points $x = \pm a$ and is subject to retention only in the case of given contact width $2a(t) = 2a$; the value of the function $a = a(t)$ is thereby determined from the equilibrium equation

$$M_0 = 2\int_0^a p(x, t)\, x dx \tag{3.74}$$

Substituting $p(x, t)$ from (3.71) into the equation (3.74) we obtain

$$\begin{aligned} M_0 = 2K(\mu)\Big\{\frac{a^2}{2}\Phi_2(\alpha, t, \alpha)\, B\Big(1-\frac{\mu}{2}, \frac{3}{2}\Big) + \\ + \int_0^a x^2 dx \int_x^a \frac{u^{\mu-2}[(1-\mu)\Phi_2(u, t, \alpha) - u\Phi_2'(u, t, \alpha)]\, du}{\sqrt{(u^2-x^2)^{\mu}}}\Big\} \end{aligned} \tag{3.75}$$

Here we used the value of the integral

$$I_3(u) = \int_0^u \frac{s^2 ds}{\sqrt{(u^2-s^2)^{\mu}}} = \frac{1}{2}\, B\Big(1-\frac{\mu}{2}, \frac{3}{2}\Big)\, u^{3-\mu}, \quad B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{3.76}$$

Changing the order of integration in the second term of relation (3.75) and using equation (3.76), we find

$$\begin{aligned} M_0 = K(\mu)\, B\Big(1-\frac{\mu}{2}, \frac{3}{2}\Big)\Big\{a^2\Phi_2(a, t, \alpha) + \\ + (1-\mu)\int_0^a u\Phi_2(u, t, \alpha)\, du - \int_0^a u^2\Phi_2'(u, t, \alpha)\, du\Big\} \end{aligned} \tag{3.77}$$

Integrating by parts the last term on the right-hand side (3.77), and then changing the order of integration, employing equations (3.52) and (3.43), we finally obtain the following equation, which relates the values $a = a(t)$ with the moment of the external forces:

$$M_0 = \frac{\sin {}^1/_2\pi\mu}{\pi(1-\mu)(2-\mu)}\int_0^a (a^2 - s^2)^{1-{}^1/_2\mu}\, \omega'(s, t)\, ds \tag{3.78}$$

Thus, when the contact width $2a(t) = 2a$ is given, then the unknown function $a = a(t)$, which enters into the solution of the Volterra equation (3.13) $\omega(x, t) = \omega^*[x, t, a(t)]$, is determined from equation (3.78).

If, however, the contact width $2a(t)$ is not given and the contact occurs along smooth surfaces, then the unknown function $a = a(t)$ is determined from the requirement that in formula (3.73) the first term, representing the solution with singularities, should vanish, i.e.

$$\Phi_2(a, t, \alpha) = \int_0^a \frac{\omega'(s, t)\, ds}{\sqrt{(a^2-s^2)^{\mu}}} = 0 \tag{3.79}$$

Consequently, when the contact width $2a(t)$ is not given, then $a = a(t)$ is determined from equation (3.79), and the unknown contact width $2a(t)$

with the aid of equation (3.78).

As an application let us consider the contact problem of pressure of a rigid die with a plane base of a given width $2a(t) = 2a$ on the half-plane under the conditions of nonlinear creep, when to the center of the die a moment M_0 is applied.

In this case, in accordance with (3.22) we shall have

$$f_0(x) = 0, \qquad F(x, t, \alpha) = \alpha^{\mu}(t)\, x^{\mu} \tag{3.80}$$

Then the solution of the Volterra integral equation (3.13) will be

$$\omega(x, t) = x^{\mu}\left(\alpha^{\mu}(t) + \int_{\tau_1}^{t} \alpha^{\mu}(\tau) R(t, \tau)\, d\tau\right) \tag{3.81}$$

where $R(t, \tau)$ is the resolvent of the creep kernel $K(t, \tau) = \partial C(t, \tau)/\partial\tau$.

From equations (3.81) and (3.79), eliminating $a^{\mu}(t)$, we find directly that the solution of equation (3.13) $\omega(x, t)$ does not depend on t and is equal to

$$\omega(x) = \frac{4M_0(1-\mu)}{\mu a^2}\, x^{\mu} \tag{3.82}$$

Substituting this expression $\omega(x)$ into (3.74) and noting that

$$\int_0^u \frac{s^{\mu-1}}{\sqrt{(u^2 - s^2)^{\mu}}} = \frac{\pi}{2\sin {}^1/_2\, \pi\mu} \tag{3.83}$$

we reduce formula (3.74) for $p(x, t)$ to the form (3.84)

$$p(x, t) = p(x) = \frac{2M_0 \pi K(\mu)(1-\mu)}{a^2 \sin {}^1/_2\, \pi\mu}\left\{\frac{a^{\mu-1}x}{\sqrt{(a^2 - x^2)^{\mu}}} + (1-\mu)\, x \int_x^a \frac{u^{\mu-2}du}{\sqrt{(u^2 - x^2)^{\mu}}}\right\}$$

where $K(\mu)$ is determined by relation (3.43).

Let us denote the second term in formula (3.84) by $(1 - \mu) I_4(x)$. We note that $I_4(x)$, being an odd function, is continuous in the whole interval $-a \leqslant x \leqslant a$, with the exception at $x = 0$ where $I_4(x)$ is discontinuous; we have

$$I_4(+0) = -I_4(-0) = \lim_{x \to 0} x \int_x^a \frac{u^{\mu-2}du}{\sqrt{(u^2 - x^2)^{\mu}}}, \qquad I_4(\pm a) = 0 \tag{3.85}$$

The integral $I_4(x)$ converges uniformly with respect to x in the interval $0 \leqslant x \leqslant a$. In fact, integrating $I_4(x)$ by parts, we obtain

$$I_4(x) = \frac{a^{\mu-3}x(a^2 - x^2)^{1-{}^1/_2\mu}}{2-\mu} + \frac{3-\mu}{2-\mu}\, x \int_x^a \frac{(u^2 - x^2)^{1-{}^1/_2\mu}}{u^{4-\mu}}\, du \tag{3.86}$$

From this the convergence of $I_4(x)$ for the values $0 \leqslant x \leqslant a$ is obvious.

For $x \to +0$ it follows directly from relation (3.85), after substitution of the integration variable $u = x/\xi$ and the conditions $\mu \leqslant 1$

$$I_4(+0) = -I_4(-0) = \int_0^1 (1-\xi^2)^{1-1/2\mu} d\xi = \frac{\pi \sqrt{\mu}}{(1-\mu) \sin {}^1/_2 \pi\mu \, \Gamma\left(\frac{1-\mu}{2}\right) \Gamma\left(\frac{\mu}{2}\right)} \tag{3.87}$$

In the following let us assume that at the point $x = 0$

$$I_4(0) = \frac{I_4(+0) + I_4(-0)}{2} = 0 \tag{3.88}$$

Formula (3.84) then takes on the form:

$$p(x) = \frac{2M_0 \pi K(\mu)(1-\mu)}{a^2 \sin {}^1/_2 \pi\mu} \left\{ \frac{a^{\mu-1} x}{\sqrt{(a^2-x^2)^\mu}} + (1-\mu) I_4(x) \right\} \quad (0 < x \leqslant a) \tag{3.89}$$

while $p(0) = 0$.

Substituting the value for $I_4(x)$ from (3.86) into (3.89), and expanding the numerator of the integral function $(u^2 - x^2)^{1-\mu/2}$ by the binomial formula, we obtain after integration

$$p(x) = \frac{2M_0 \pi K(\mu)(1-\mu)}{a^2 \sin {}^1/_2 \pi\mu} \left\{ \frac{a^{\mu-1} x}{\sqrt{(a^2-x^2)^\mu}} + \frac{(1-\mu) a^{\mu-3} x (a^2-x^2)^{1-1/2\mu}}{2-\mu} + \right.$$
$$\left. + \frac{(3-\mu)(1-\mu)}{2-\mu} \sum_{k=1}^{k=\infty} \frac{\Gamma\left(K-2+\frac{\mu}{2}\right)}{(2K-1)\Gamma(K)\Gamma\left(\frac{\mu}{2}-1\right)} \left[1 - \left(\frac{x}{a}\right)^{2k-1}\right] \right\} \tag{3.90}$$

Substituting the value $K(\mu)$ from (3.43) into (3.91), we obtain the final formula for the determination of the pressure $p(x)$ on the contact area underneath the die:

$$p(x) = \frac{2M_0 \Gamma\left(\frac{3-\mu}{2}\right) \Gamma\left(\frac{\mu}{2}\right) \sin \frac{\pi\mu}{2}}{a^2 \pi \sqrt{\pi}} \left\{ \frac{a^{\mu-1} x}{\sqrt{(a^2-x^2)^\mu}} + \frac{(1-\mu) a^{\mu-3} x (a^2-x^2)^{1-1/2\mu}}{2-\mu} + \right.$$
$$\left. + \frac{(3-\mu)(1-\mu)}{(2-\mu)} \sum_{k=1}^{k=\infty} \frac{\Gamma\left(k-2+\frac{\mu}{2}\right)}{(2k-1)\Gamma(k)\Gamma\left(\frac{\mu}{2}-1\right)} \left[1 - \left(\frac{x}{a}\right)^{2k-1}\right] \right\} \tag{3.91}$$

where $p(0) = 0$ and $p(-x) = -p(x)$.

It follows from the solution (3.91) that also in this case the creep of the material of the compressed bodies does not influence the law of contact stress distribution $p(x)$, since the contact between these bodies is along a straight line.

For $\mu = 1$, i.e. under the conditions of linear creep, the formula for $p(x)$ (3.91) reduces to

$$p(x) = \frac{2M_0}{\pi a^2} \frac{x}{\sqrt{a^2 - x^2}} \tag{3.93}$$

which coincides with the known solution [)1,17] of the contact problem of the linear theory of creep or the linear theory of elasticity (in the present case, they coincide identically) for a plane die of width $2a$, when a moment M_0 is applied at its center.

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